Some explanations on convolution

We define convolution in the continuous case as:

$$\underbrace{(f * g)}_{\text{a new function}}(y) := \int_{\mathbb{R}} f(x)\, g(y-x)\, dx \qquad \text{1D}$$

$$= \int_{\mathbb{R}} f(y-x)\, g(x)\, dx$$

$$(f * g)(x,y) = \int_{\mathbb{R}^2} f(s,t)\, g(x-s, y-t)\, ds\, dt \qquad \text{2D}$$

$$= \int_{\mathbb{R}^2} f(x-s, y-t)\, f(s,t)\, ds\, dt$$

If f and g are periodic functions with period $\underline{T}$, we define the <u>periodic</u> convolution

$$(f * g)(x) = \int_0^T f(y)\, g(x-y)\, dy, \quad 0 \le x \le T. \qquad \text{1D}$$

$$(f * g)(x,y) = \int_0^T \int_0^T f(s,t)\, g(x-s, y-t)\, ds\, dt,$$

$$0 \le x, y \le T \qquad \text{2D}$$

As a result, we define the following

Suppose f and g are defined on $\mathbb{Z}$ (integers), then the discrete convolution is defined by:

$$(f * g)(n) := \sum_{m=-\infty}^{\infty} f(m)\, g(n-m) \qquad \text{1D}$$

$$(f * g)(m, n) = \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} f(s, t)\, g(m-s, n-t). \qquad \text{2D}$$

$$\left(f(m-s, n-t)\, g(s, t) \right)$$

Example:

$$\text{mean filter} \quad K = \frac{1}{9}\begin{array}{ccc} & 0\ \ 0\ \ 0\ \ 0 & \\ \begin{matrix}0\\0\\0\end{matrix} & \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} & \begin{matrix}0\\0\\0\end{matrix} \\ & 0\ \ 0\ \ 0 & \end{array}$$

Let A be an $M \times N$ image,

What is the effect of $A * K$?

(All entries not defined are zero,

Also, we require

$$A \quad \begin{array}{c|c} & 0 \quad 1 \cdots N-1 \\ \begin{matrix}0\\1\\ \vdots \\ M-1\end{matrix} & \begin{pmatrix} \quad & & \quad \\ & & \\ & & \\ & & \end{pmatrix} \end{array} \qquad K \quad \begin{array}{c|c} & -1 \quad 0 \quad 1 \\ \begin{matrix}-1\\0\\1\end{matrix} & \begin{pmatrix} \quad & & \quad \\ & & \\ & & \end{pmatrix} \end{array}$$

When f and g are periodic with period N, we can define the following circular discrete convolution:

$$f * g(n) = \sum_{m=0}^{N-1} f(n)\, g(m-n) = \sum_{m=0}^{N-1} f(m-n)\, g(n) \quad (0 \le n \le N-1) \quad \text{1D}$$

$$f * g(m,n) = \sum_{s=0}^{N-1} \sum_{t=0}^{N-1} f(s,t)\, g(m-s, n-t) = \sum_{s=0}^{N-1} \sum_{t=0}^{N-1} f(m-s, n-t)\, g(s,t). \quad \text{2D}$$

$$(0 \le m, n \le N-1)$$

(The circular convolution is closely related to DFT and FFT.
When indicated that two images are periodically extended, use this definition)

Convolutions in convolutional neural networks (CNN)

(More straightforward)

$$\underset{\text{new image}}{g} = \underset{\text{original image}}{f} * \underset{\text{convolution kernel}}{K}$$

f: $N \times M$ image

K: $\underbrace{S \times T}_{\text{usually odd}}$ kernel matrix

g: (usually) $N \times M$ image

$$\begin{array}{ccccccc}
f(0,0) & f(0,1) & f(0,2) & & & & \\
f(1,0) & f(1,1) & f(1,2) & f(1,3) & \cdots & f(1,M) \\
f(2,0) & f(2,1) & f(2,2) & f(2,3) & \cdots & f(2,M) \\
 & f(3,1) & f(3,2) & f(3,3) & \cdots & f(3,M) \\
 & \vdots & & \ddots & & \vdots \\
 & f(N,1) & f(N,2) & f(N,3) & \cdots & f(N,M)
\end{array}$$

K $\longrightarrow$ output $g(1,1)$

If K is a 3×3 kernel,

then $g(\alpha, \beta) =$

$$f(\alpha-1, \beta-1)\, g(1,1) + f(\alpha-1, \beta)\, g(1,2) + f(\alpha-1, \beta+1)\, g(1,3)$$

$$+ f(\alpha, \beta-1)\, g(2,1) + f(\alpha, \beta)\, g(2,2) + f(\alpha, \beta+1)\, g(2,3)$$

$$+ f(\alpha+1, \beta-1)\, g(3,1) + f(\alpha+1, \beta)\, g(3,2) + f(\alpha+1, \beta+1)\, g(3,3)$$

for $1 \leq \alpha \leq N$, $1 \leq \beta \leq M$.

We assume one of the followings.

① f periodically extended,

i.e., $f(\alpha, \beta) = f(\alpha + sN, \beta + tM)$,

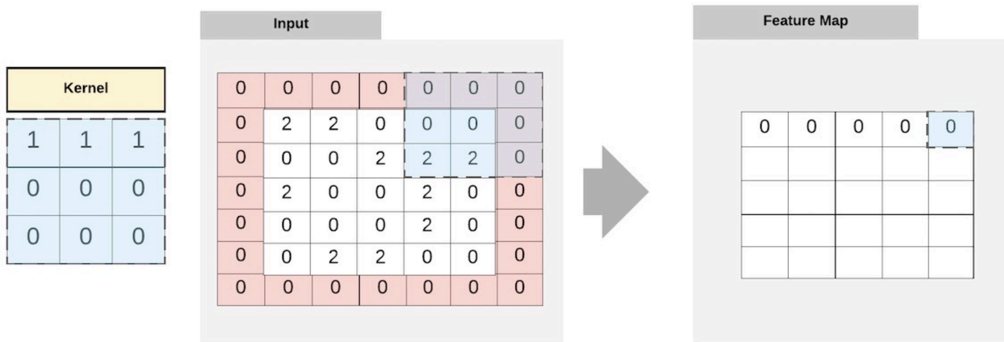
for any $s, t \in \mathbb{Z}$, $1 \leq \alpha \leq N$, $1 \leq \beta \leq M$.

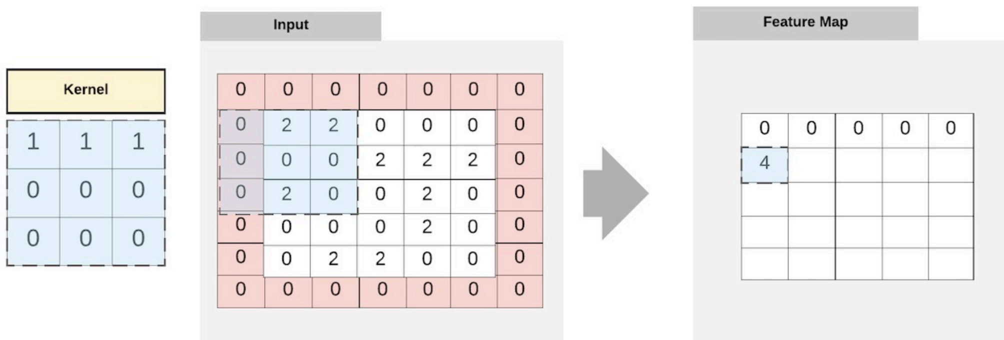
② $f(x, y) = 0$ for $(x, y) \notin [\![1, N]\!] \times [\![1, M]\!]$.

(zero padding) most common choice

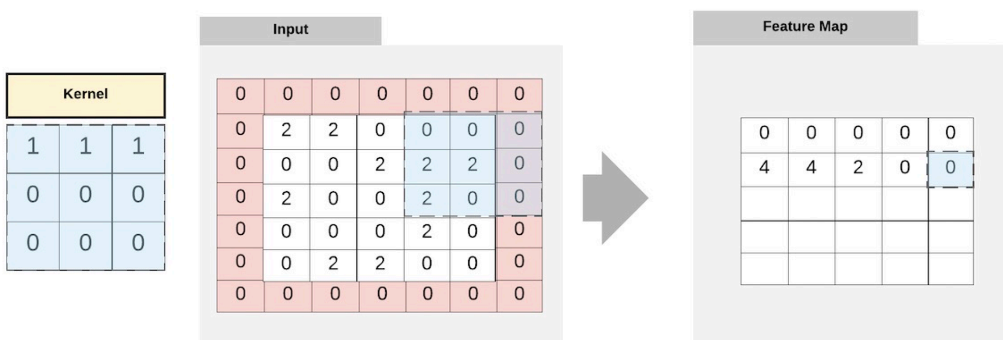
③ $f(x, y) = c(x, y)$ for some function c

for $(x, y) \notin [\![1, N]\!] \times [\![1, M]\!]$.

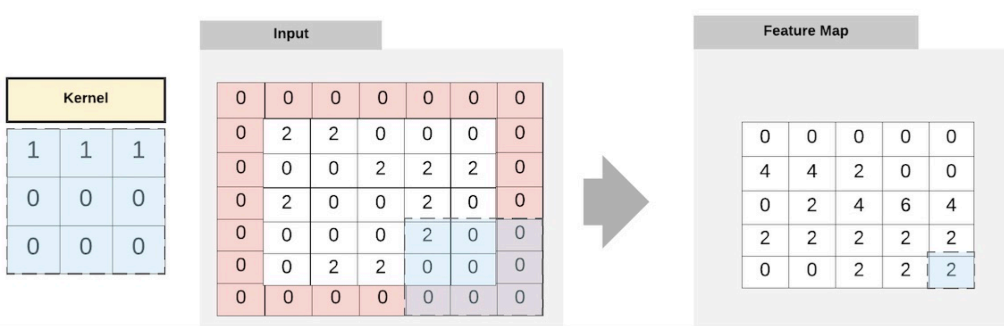




SAME 5 SETS OF (1) ELEMENT-WISE MULTIPLICATION (2) SUMMATION



REPEAT FOR REMAINING ROWS



Remark:

Later on, when you see convolutions in image processing in the spatial domain, you can understand convolutions in this way if you think doing so help you understand the course materials.

Haar and Walsh transform

Why do we need orthogonality of basis functions?

(you will be asked to prove othogonality in HW2)

Some idea:

We consider vector decomposition in an inner product space. For simplicity, we restrict to the finite dimensional case.

(For Haar / Walsh, we consider the inner product space $(L^2(\mathbb{R}), \langle \cdot \rangle)$, $\langle f, g \rangle = \int_{\mathbb{R}} fg$, which will also be introduced in HW2)

If $\{f_1, \cdots, f_n\}$ form an orthonormal basis, i.e., $\|f_i\| = 1$, $\langle f_i, f_j \rangle = 0$ for $i \neq j$, for an arbitrary f, suppose $f = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n$.

Note $\langle f, f_i \rangle = \langle a_1 f_1 + a_2 f_2 + \cdots + a_n f_n, f_i \rangle$

$\overset{\parallel}{\int_{\mathbb{R}} f f_i}$ $= a_i \langle f_i, f_i \rangle = a_i$

The representation is unique:

assume $f = a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = b_1 f_1 + b_2 f_2 + \cdots + b_n f_n$.

$\Rightarrow (a_1 - b_1) f_1 + (a_2 - b_2) f_2 + \cdots + (a_n - b_n) f_n = 0.$

$\Rightarrow \langle (a_1 - b_1) f_1 + \cdots + (a_n - b_n) f_n, (a_1 - b_1) f_1 + \cdots + (a_n - b_n) f_n \rangle$

$= (a_1 - b_1)^2 \overset{1}{\langle f_1, f_1 \rangle} + (a_1 - b_1)^2 \overset{0}{\langle f_1, f_2 \rangle} + \cdots + (a_1 - b_1)^2 \overset{0}{\langle f_1, f_n \rangle}$

$+ (a_2 - b_2)^2 \overset{0}{\langle f_2, f_1 \rangle} + (a_2 - b_2)^2 \overset{1}{\langle f_2, f_2 \rangle} + \cdots + (a_2 - b_2)^2 \overset{0}{\langle f_2, f_n \rangle}$

$+ \cdots$

$+ (a_n - b_n)^2 \overset{0}{\langle f_n, f_1 \rangle} + \cdots + (a_n - b_n)^2 \overset{1}{\langle f_n, f_n \rangle}$

$= (a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2$

$= 0$

$\Rightarrow a_i = b_i$ for any i